LINEAR INCOMPRESSIBLE WAVES IN SOLAR PERIODIC STRUCTURES

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Abstract
The spatial structuring of solar and space plasmas is known to have a dispersive effect on incident waves. Many solar features posses a periodic structure with structures having alternating properties. This study analyses the effect of periodic alternation of magnetic slabs on wave propagation using the Bendickson & Dowling model (1996). We have obtained the criteria for the appearance of standing waves taking into account the spatial scaling of the system and the strength of the magnetic field. We have obtained the correlation between the observed standing waves and the number of magnetic slabs in plume/interplume regions.

1. Introduction

The recent observations revealed the existence of waves and oscillations in the solar atmosphere.
Waves are responsible for carrying energy and momentum, creating instabilities, generating phenomena like magnetic reconnection, phase mixting etc. They can also serve as unique tool for plasma diagnostics due to their capability to carry information about the medium in which they propagate.

Many solar features (plume/interplume region) show a transversal periodicity Marcu (2005) . Waves propagating along magnetic structures with a transversal periodicity are likely to “feel” the effect of this periodicity structuring provided its transversal wavelength is of the same order as the transversal spatial organization scale. The possibility of wave propagation in periodic structures has been discussed by Hollweg (1982), Berton & Heyvaerts (1987), Uralov (2003) in the linear regime. Recently, Diaz &Roberts (2006)studied the propagation of fast MHD oscillations in periodic structures, modeling prominence fibrils and Marcu et al. (2006) have extended the the study of Berton & Heyvaerts (1987) by including the steady motion .

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The aim of the present paper is to investigate linear wave propagation in a magnetically periodic structure with N – finite - unit cells (magnetic slabs)(Figure 1). The global dispersive effect of N such unit cells can be obtained by using the transmission coefficient and the transmittance of one unit cell (Bendickson and Dowling model) and Bloch's theory..

In section 2 the dispersion relation for a unit cell is derived and in section 3 we will apply Bendickson and Dowling model in the special case of a plasma embedded in a structured magnetic field. In section 4 the we have studied the propagation of linear incompressible waves while in the last section the model is applied to the coronal plume/interplume region.

2. Derivation of the dispersion relation

We consider an ideal, perfectly conducting fluid permeated by a magnetic field of constant direction along the z-axis and periodic along the x-axis. Supposing that the wavelengths are smaller than the gravitational scale –height, the gravitational effects are neglected. We suppose that the medium consists of alternating magnetic slabs (with widths L_i and L_e) with a homogeneous magnetic field inside them (B_i and B_e) and a sharp discontinuity at the boundary, as shown in Figure 1. Denote $L = L_i + L_e$ the transversal length of the unit cell and ρ_i and ρ_e the (homogeneous) densities in the two regions (Marcu at al. 2006).

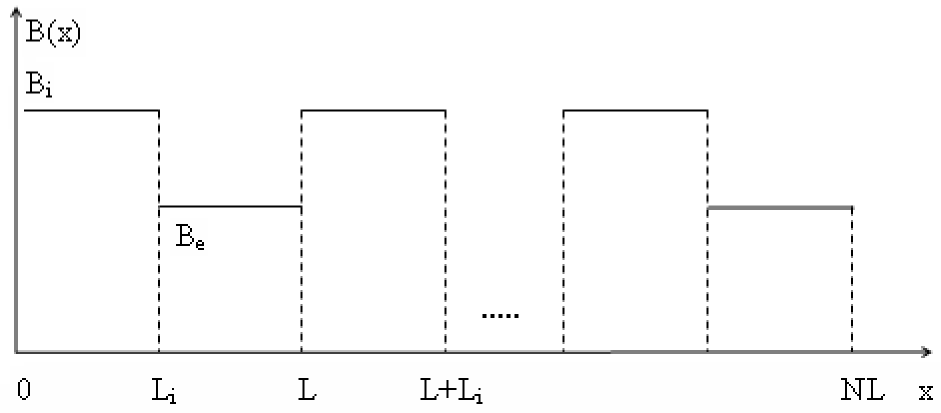

Fig. 1. Profile of the magnetic field in a periodic equilibrium state

The continuity of the total pressure at each boundary can be expressed by

$$\frac{d}{dx}(p_0 + \frac{B_0^2}{2\mu}) = 0, \tag{1}$$

and the density contrast in the two adjacent regions is described by

$$R_\rho = \frac{\rho_i}{\rho_e} = \frac{2c_{S_e}^2 + \gamma c_{A_e}^2}{2c_{S_i}^2 + \gamma c_{A_i}^2}, \tag{2}$$

where $c_{S_{i,e}}$ and $c_{A_{i,e}}$ are the sound and Alfven speeds in the two layers, γ the adiabatic index and μ the magnetic permeability.

We perturb the system and write all physical quantities in the form $f_0 + f$, where f_0 are the equilibrium values and f their Eulerian perturbations; we suppose small but finite deviation from the equilibrium value, i.e. linear waves only. In both regions the plasma dynamics is described by the system of linearized ideal MHD equation (Roberts, 1981). Since the equilibrium quantities depend on x only, we can write the perturbations as $f = \hat{f}(x)e^{i(\omega t - k_y y - k_z z)}$. Assuming $\hat{v}_y = \hat{b}_y = 0$ the velocity perturbation describing compressional waves equation is (Roberts 1981)

$$\frac{d^2\hat{v}_x}{dx^2} - (q^2 + k_y^2)\hat{v}_x = 0 \tag{3}$$

where $$q^2(x) = \frac{(k_z^2 c_A^2 - \omega^2)(k_z^2 c_S^2 - \omega^2)}{(k_z^2 c_T^2 - \omega^2)(c_A^2 + c_S^2)},$$

$c_{T_{i,e}} = c_{S_{i,e}} c_{A_{i,e}} / (c_{S_{i,e}}^2 + c_{A_{i,e}}^2)^{1/2}$ being the tube (cusp) velocity.

In the case of $k_y = 0$ the solutions of Eq. (3) inside and outside the slab can be written as

$$\hat{v}_{xi} = \alpha_i e^{q_i x} + \beta_i e^{-q_i x} \tag{4.1}$$

$$\hat{v}_{xe} = \alpha_e e^{q_e x} + \beta_e e^{-q_e x} \tag{4.2}$$

Applying the continuity conditions for the normal component of velocity, $\hat{v}_x$ and total pressure at the boundaries of two adjacent regions and using Bloch's theorem, the dispersion relation for linear compressional waves, for body modes, is (Marcu et al. 2006)

$$\cos(K_0 L) = \cos(q_i L_i)\cos(q_e L_e) + \frac{1}{2}\left(S - \frac{1}{S}\right)\sin(q_i L_i)\sin(q_e L_e) \tag{5}$$

where K_0 is the Bloch wavenumber, $L = L_i + L_e$, q_i and q_e are the effective wavenumbers for each of the two mediums,

$$q_{i,e}^2 = \frac{(k_z^2 c_{A_{i,e}}^2 - \omega^2)(k_z^2 c_{S_{i,e}}^2 - \omega^2)}{(k_z^2 c_{T_{i,e}}^2 - \omega^2)(c_{A_{i,e}}^2 + c_{S_{i,e}}^2)}; \quad S = \frac{q_i}{q_e}\frac{\rho_i}{\rho_e}\frac{(k_z^2 c_{S_i}^2 - \omega^2)(k_z^2 c_{T_e}^2 - \omega^2)}{(k_z^2 c_{S_e}^2 - \omega^2)(k_z^2 c_{T_i}^2 - \omega^2)}$$

3. Group velocity for a N periodic potential

The complex transmission coefficient, t (containing phase information) for a unit cell is defined as the ratio of the transmitted wave to the incident wave.

$$t(\omega) = \frac{v_x(L)}{v_x(0)} = e^{iK_0 L}, \quad t = a + ib\,; \tag{6}$$

where $a = \mathrm{Re}\{t\} = \cos(K_0 L)$ Using (5), the real part of the transmission coefficient can be written as

$$a = \mathrm{Re}\{t\} = \cos(q_i L_i)\cos(q_e L_e) + \frac{1}{2}(S - \frac{1}{S})\sin(q_i L_i)\sin(q_e L_e) \tag{7}$$

The total phase, φ, accumulated by the wave while going through the unit cell is

$$\varphi = k_z L\,, \tag{8}$$

$$\tan\varphi = \frac{b}{a} = \tan(k_z L) \tag{9}$$

which yields

$$b = \mathrm{Im}\{t\} = a\tan(k_z L) = \cos(K_0 L)\tan(k_z L) \tag{11}$$

Introducing Eqs.(8)-(11) in (6) the transmission coefficient yields

$$t = \cos(K_0 L)[1 + i\tan(k_z L)] \tag{12}$$

Using Eq.(12) the transmittance of one unit cell is given by

$$T = |t|^2 = \frac{\cos^2(K_0 L)}{\cos^2(k_z L)} \tag{13}$$

Bendickson and Dowling 1996 demonstrated that the group speed depends only of the characteristics of one unit cell, the total number N of unit cells and the Bloch phase $\beta = K_0 L$. The characteristics of one cell are given by the complex transmission coefficient and transmittance.

The general formula for the group speed for a N period structure is

$$V_g^N(\omega) = \frac{NL\{(\cos^2 N\beta) + \eta^2[\sin(N\beta)/\sin\beta]^2\}}{(1/2)[\sin(2N\beta)/\sin\beta][\eta' + \eta\xi\xi'/(1-\xi^2)] - N\eta\xi'/(1-\xi^2)} \tag{14}$$

where $\beta = K_0 L$,

$$\eta = \frac{b}{T} = \frac{\mathrm{Im}\{t\}}{T} = \frac{\sin(2k_z L)}{2\cos(K_0 L)}, \eta' = -\frac{1}{2}\frac{1}{\cos^2(K_0 L)}\sin(2k_z L)\psi\,,$$

$$\xi = \frac{a}{T} = \frac{\mathrm{Re}\{t\}}{T} = \frac{\cos^2(k_z L)}{\cos(K_0 L)},\ \xi' = -\frac{1}{\cos^2(K_0 L)}\psi \text{ and } \psi = d(RHS(5))/d\omega.$$

The equation (14) can be applied to a periodical magnetic structure without loss of accuracy, where the changes in magnetic field strength represent the potential leaps met by the waves while they propagate. The nature of the medium leads to magnetoacoustic waves. This does not restrict the generality of (14), since it does not explicitly depend on the nature of the propagating modes, but on the cell properties; these properties can be defined for any mode, as it was done in (6)-(13).

4. Incompressible modes

This section presents a method for obtaining the relationship between the velocity amplitude of the perturbation, the frequency of the incident wave and the number of the slabs for magnetoacoustic modes propagating in an incompressible plasma, permeated by a periodic field.

The incompressible plasma approximation $c_{s_{i,e}} \to \infty$, leads to

$$\left|q_{i,e}\right| = k_z, \quad S = R_\rho \frac{k_z{}^2 c_{A_i}{}^2 - \omega^2}{k_z{}^2 c_{A_e}{}^2 - \omega^2} \tag{15}$$

By introducing the dimensionless arguments $\theta = k_z L_i$ and $R_l = \frac{L_i}{L_e}$, $c = \frac{\omega}{k_z}$ the phase speed of the waves and considering $R_\rho = 1$, the group velocity in this magnetic structured medium becomes

$$V_g = \frac{4N(1+\frac{1}{R_l})(c^2 - c^2_{A_e})^2 (c^2 - c^2_{A_i})^2 \theta \cos^2\beta \csc\theta \csc(2\Theta) \csc(\frac{\theta}{R_l}) \sin\beta \{\cos(\beta N) + \csc^2(2\beta) \sin^2(\beta N) \sin^2(2\Theta)\}}{c(c^2_{A_e} - c^2_{A_i})[2c^2 + c^4_{A_e} + c^4_{A_i} - 2c^2(c^2_{A_i} + c^2_{A_e})]\{4N\csc(2\beta) - [1 + 4\cos^2\Theta \csc^2(2\beta)]\sin(2\beta N)\}} \tag{16}$$

where $\Theta = (1 + \frac{1}{R_l})\theta$.

The dispersion equation (5) for an incompressible medium becomes

$$\cos(K_0 L) =$$

$$\cos(\theta)\cos(\frac{\theta}{R_l}) + \frac{1}{2}(R_\rho \frac{k_z{}^2 c_{A_i}{}^2 - \omega^2}{k_z{}^2 c_{A_e}{}^2 - \omega^2} - \frac{k_z{}^2 c_{A_e}{}^2 - \omega^2}{R_\rho (k_z{}^2 c_{A_i}{}^2 - \omega^2)}) \sin(\theta) \sin(\frac{\theta}{R_l}) \tag{17}$$

If we note $U = \dfrac{\cos(K_0 L) - \cos(\theta)\cos(\frac{\theta}{R_l})}{\sin(\theta)\sin(\frac{\theta}{R_l})}$ and take into account equation (15) results

$$S^2 - 2US - 1 = 0$$
$$S_{\pm} = U \pm \sqrt{U^2 + 1}$$

where S_+ stands for sausage and S_- stands for kink. The S_+ solution can be expressed as

$$R_\rho (k_z{}^2 c_{A_i}{}^2 - \omega^2) - (U \pm \sqrt{U^2 + 1})(k_z{}^2 c_{A_e}{}^2 - \omega^2) = 0 \quad (18)$$

By solving equation (18) we can find the relation between ω and k_z

$$\omega = k_z \frac{\sqrt{-R_\rho v_{A_i}^2 - v_{A_e}^2 T + v_{A_e}^2 \sqrt{1 + (-T)^2}}}{\sqrt{-R_\rho - T + \sqrt{1 + (-T)^2}}} \quad (19)$$

where

$$T = \left(\cos(\theta)\cos(\frac{\theta}{R_l}) - \cos(K_0 L) \right) / \left(\sin(\theta)\sin(\frac{\theta}{R_l}) \right)$$

Because ω must be real the relation between the spatial scaling and the wave characteristics is

$$R_l > \frac{\theta}{\arccos\left(\frac{\cos(K_0 L)}{\cos\theta} \right)} \quad (20)$$

Taking the solution (19) and rewrite equation (15) with it gives V_g^N as a function of the wavenumber k_z.

5. Plume/interplume region

In order to study the possible modes that can appear in the periodic structures of the plume region a numerical treatment will we used. The characteristic speeds in the plume region are $c_{A_i} = 750 km/s$ and $c_{A_e} = 1350 km/s$. According to the model presented in section 4 the solutions of the dispersion relation for an incompressible medium are calculated and substituted into the group velocity relation. The expression obtained is used to compute the group speed dependence of the wavenumber , with the following numerical substitutions $R_\rho = 1; K_0 L = \dfrac{\pi}{4}$.

The group speed is plotted as a function of k and N for sausage modes (Figure 2) and kink modes (Figure 3). In all the numerical representation the values given to the physical parameters satisfy condition (20).

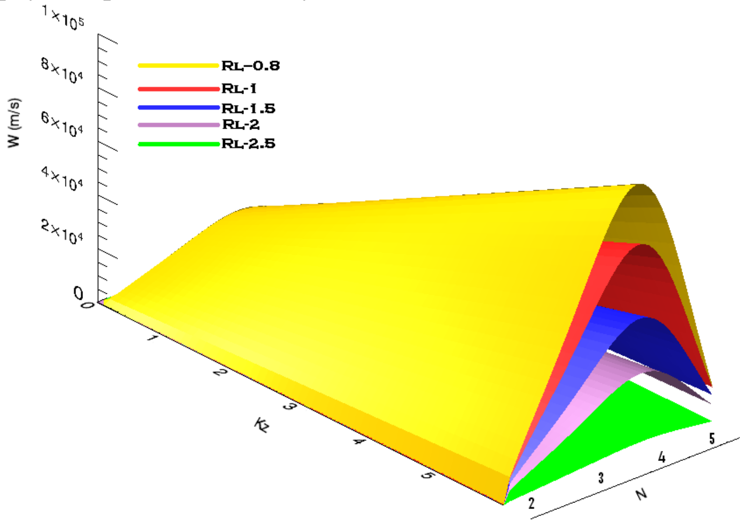

Fig. 2. The group speed for the sausage mode as a function of k and N

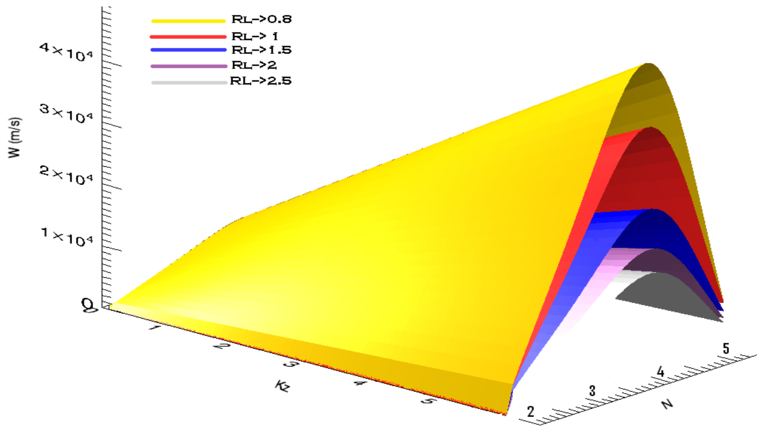

Fig. 3. The group speed for the kink mode as a function of k and N

An observation can be made concerning the numerical value of the group speeds for the two modes. In the case of identical physical parameters the group speed of the sausage mode is bigger than in the case of the kink mode. This was represented in Figure 4 for relative lengths $R_l = 0.8$ and $R_l = 1.0$

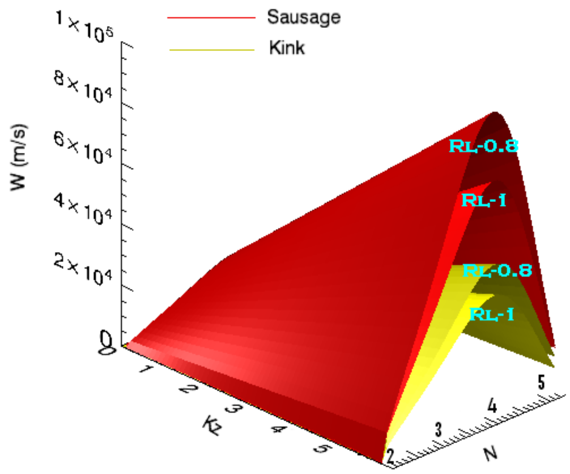

Fig. 4. Comparison between the group speed for the sausage mode (red) and the kink

mode (yellow) for $R_l = 0.8$ and $R_l = 1.0$

Conclusions

Standing waves can be observed and analytical and numerical calculations correlate them with the number N (for N smaller than 10) of slabs in the medium. A maximum for the value of the group speed is reached for the minimal value of the relative length, $R_l = 0.8$, in the case of the sausage mode. The model is accurate only for narrow frequency bands of the incident wave. A correct model for wide frequency of the incident wave as the ones observed with SOHO and TRACE will be developed in a future article.

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